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A Property of Chebyshev Polynomials

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We shall prove the following.

THEOREM. *If P is a polynomial of degree n with n distinct zeros in* $[-1, 1]$ *and*

$$| P(\cos(k\pi/n)| = 1, \qquad k = 0, 1, \ldots, n, \tag{1}$$

then either $P(x) = T_n(x)$ or $P(x) = -T_n(x)$, where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of degree n.

This theorem answers affirmatively a problem posed by C. Micchelli and T. Rivlin at the conference on "Linear Operators and Approximation" held in Oberwolfach in the summer of 1971, (see [1, p. 498]).

For the proof, we will use a lemma due to W. W. Rogosinski [2]. Throughout, we assume that P is a polynomial of degree n with n distinct zeros in $[-1, 1]$, satisfying (1).

LEMMA 1. (Rogosinski [2]). *If* $P(x) = a(x - x_1) \cdots (x - x_n)$, *then* $| a | \leqslant 2^{n-1}$.

Proof of Theorem. We wish to show that if $P(x) = a(x - x_1) \cdots (x - x_n)$, then, $| a | > 2^{n-1}$, or $P = \pm T_n$. This coupled with Lemma 1 proves the theorem. We expand P in terms of Chebyshev polynomials as

$$P(x) = \sum_0^n \lambda_k T_k(x).$$

Since the coefficient of x^n in T_n is 2^{n-1}, we have $\lambda_n 2^{n-1} = a$. Now,

$$P(\cos\theta) = \sum_0^n \lambda_k \cos k\theta = 1/2 \sum_0^n \lambda_k(e^{ik\theta} + e^{-ik\theta}).$$

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This leads us to consider the polynomial

$$R(z) = \tfrac{1}{2} z^n \sum_0^n \lambda_k(z^k + z^{-k}).$$

At each $2n$th root of unity $e^{ik\pi/n}$, $| R(e^{ik\pi/n})| = 1$, $k = 1,\ldots, 2n$. Also, R has all its zeros on the unit circle, namely at the points $z_1, \ldots, z_{2n}$, where z_k is that point on the unit circle with $\mathrm{Re}(z_k) = x_k$ and $\mathrm{Im}(z_k) > 0$, $k = 1,\ldots, n$ and $z_{k+n} = \bar{z}_k$, $k = 1,\ldots, n$. Hence,

$$R(z) = \tfrac{1}{2}\lambda_n(z - z_1) \cdots (z - z_{2n}).$$

The polynomial $z^{2n} - 1$, vanishes at each of the $2n$th roots of unity and so

$$z^{2n} - 1 = (z - e^{i\pi/n}) \cdots (z - e^{i2n\pi/n}).$$

This gives

$$1 = \prod_{k=1}^{2n} | R(e^{ik\pi/n})| = \left(\frac{|\lambda_n|}{2}\right)^{2n} \prod_{j,k=1}^{2n} | e^{ik\pi/n} - z_j |$$

$$= \left(\frac{|\lambda_n|}{2}\right)^{2n} \prod_{j=1}^{2n} | z_j^{2n} - 1 | \leqslant | \lambda_n |^{2n}. \qquad (2)$$

The last inequality is strict unless each term $z_j^{2n} - 1$ is equal to -2. One checks easily that this would imply that either $P = T_n$ or $P = -T_n$. Thus, if $P \neq \pm T_n$, then (2) shows that

$$1 < | \lambda_n | = 2^{-n+1} | a |,$$

as desired.

Remark. Our original proof did not use Lemma 1. This lemma was kindly pointed out to us by Michelli and Rivlin and this considerably simplified our original proof.

References

1. Linear Operators and Approximation, Proceedings of the conference held in Oberwolfach, *ISNM*, **20**, Birkhäuser, Basel, 1972.
2. W. W. Rogosinski, Some elementary inequalities for polynomials, *Math. Gaz.* **39** (1955), 7–12.